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A continuous spectral density for a random field of continuous-index

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ABSTRACT

Linear dependence coefficients are defined for random fields of continuous-index, which are modified from those already defined for random fields indexed by an integer lattice. When a selection of these linear dependence conditions are satisfied, the random field will have a continuous spectral density function. Showing this involves the construction of a special class of random fields using a standard Poisson process and the original random field.

1. Introduction and definitions

When considering a stochastic process or time series, the covariance function (sometimes referred to as the autocovariance function) is used to study the pattern of the process as it moves through the index set, which usually is time. When the spectral density function exists, it is the Fourier transform of the covariance function. It helps when studying frequency properties of the process [1]. The continuity and positivity of the spectral density function is closely connected with certain linear dependence coefficients, and plays a significant role in the spectral density estimation. This article will focus mainly on the two linear dependence coefficients $r'(n)$ and $\zeta(n)$. They are defined for discrete-indexed random fields in both [3,4], and will be defined in this article for random fields having a continous index. A random field is a stochastic process whose index set is multi-dimensional.

In 1991, Bradley [5] proved that a discrete-indexed random field has a continuous spectral density function if $r^*(n) \to 0$. The coefficient $r^*(n)$ uses Euclidean distance between random variables while $r'(n)$ uses a distance of n in at least one of the dimensions. In 1997, Curtis Miller [2] did work in finding a continuous spectral density function for a random field with continuous index assuming $\rho^*(n) \to 0$ (similar to but stronger than $r^*(n) \to 0$) and another condition. A definition of the ρ^* coefficient can be found in both [7,8]. In 2000, Bradley [4] showed that a discrete-indexed random field has a continuous spectral density function assuming only $\zeta(n) \to 0$. This article will show that a random field of continuous index has a continuous spectral density assuming $\zeta(n) \to 0$, $r'(a) < 1$ for some a, and another condition.

The setting of this article will be on a probability space $(\Omega, \mathcal{F}, P)$, in which Ω is the sample space, $\mathcal{F}$ is a σ-field on Ω, and P is a probability measure on $(\Omega, \mathcal{F})$. A random variable X is a real or complex valued $\mathcal{F}$-measurable function defined on Ω. A random field is usually denoted by $(X_t : t \in V^d)$ where V is either $\mathbb{Z}$ or $\mathbb{R}$ and d is a positive integer. For a random field $(X_\nu : \nu \in \mathbb{R}^d)$ on a probability space $(\Omega, \mathcal{F}, P)$, it is understood that the function $(\nu, \omega) \mapsto X_\nu(\omega)$ for $(\nu, \omega) \in \mathbb{R}^d \times \Omega$ is measurable with respect to the product σ-field $\mathcal{R}^d \times \mathcal{F}$ where $\mathcal{R}^d$ is the Borel σ-field on $\mathbb{R}^d$.

For the definition of weakly stationary below, let $k := (k_1, k_2, \ldots, k_d) \in \mathbb{R}^d$, $\ell := (\ell_1, \ell_2, \ldots, \ell_d) \in \mathbb{R}^d$, and $k - \ell := (k_1 - \ell_1, k_2 - \ell_2, \ldots, k_d - \ell_d)$.

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Definition 1.1. Let d be a positive integer, and V be either $\mathbb{Z}$ or $\mathbb{R}$. A complex valued random field $X := (X_k : k \in V^d)$ is weakly stationary if it has the following three properties:

(1) $E|X_k|^2 < \infty$ for all $k \in V^d$.
(2) There exists an $m \in \mathbb{C}$ such that $EX_k = m$ for all $k \in V^d$.
(3) There exists a function $\gamma : V^d \to \mathbb{C}$ such that for every $k, \ell \in V^d$, $E(X_k - m)(\overline{X_\ell - m}) = \gamma(k - \ell)$.

In the setting of Definition 1.1, a complex valued random field is one such that for each $k \in V^d$, $X_k \in \mathbb{C}$. In addition, if $m = 0$, then the random field is also centered and will be called *CCWS* (centered, complex, and weakly stationary). The function γ will be referred to as the *covariance function*.

Since it is assumed that $(\nu, \omega) \mapsto X_\nu(\omega)$ is measurable with respect to the product σ-field $\mathcal{R}^d \times \mathcal{F}$, γ is continuous at the origin from lines 13–16 on page 60, and lines 10–12 of section 3 on page 518 of [6]. If a random field X is CCWS, it turns out that γ is uniformly continuous over all $\mathbb{R}^d$. The following will show this.

Suppose that X is a CCWS random field. Using the fact that γ is continuous at the origin, fix $\varepsilon > 0$ and let $\delta > 0$ be such that $\|\nu\| < \delta$ implies that $|\gamma(\nu) - \gamma(0)| < \varepsilon/2$. Then for all $\nu, r \in \mathbb{R}^d$,

$$\begin{aligned} E|X_\nu - X_r|^2 &= \left(EX_\nu\overline{X_\nu} - EX_\nu\overline{X_r} - EX_r\overline{X_\nu} + EX_r\overline{X_r}\right) \\ &= \left(\gamma(0) - \gamma(\nu - r) - \gamma(r - \nu) + \gamma(0)\right). \end{aligned}$$

Thus, whenever $\|\nu - r\| < \delta$, one has that $E|X_\nu - X_r|^2 < \varepsilon$.

If X is degenerate, then the covariance function is the constant function 0, and therefore, uniformly continuous. When X is non-degenerate, $\|X_0\|_2 > 0$. Choose $\varepsilon > 0$ arbitrarily. Using the previous argument and the fact that $\|X_0\|_2 < \infty$ by weak stationarity (Definition 1.1), let $\delta > 0$ be small enough so that if $\|\nu - r\| < \delta$, then $\|X_\nu - X_r\|_2 < \varepsilon/\|X_0\|_2$. Then, for all ν, $r \in \mathbb{R}^d$ such that $\|\nu - r\| < \delta$,

$$\begin{aligned} |\gamma(\nu) - \gamma(r)| &= |EX_\nu\overline{X}_0 - EX_r\overline{X}_0| \\ &= |E(X_\nu - X_r)\overline{X}_0| \\ &\le \|X_\nu - X_r\|_2 \cdot \|X_0\|_2 \\ &< \varepsilon. \end{aligned}$$

Thus, the complex covariance function γ is uniformly continuous on $\mathbb{R}^d$.

In the continuous-index case, the spectral density function is defined over all $\mathbb{R}^d$. In this context, $\mathrm{d}m_d(x)$ will be understood as $(2\pi)^{-d}\mathrm{d}x$ where $\mathrm{d}x$ denotes Lebesgue measure on $\mathbb{R}^d$. This is in a spirit similar to that of the discrete-index case when the spectral density is defined on the d-dimensional unit circle.

Definition 1.2. A Borel measurable, non-negative integrable function f on $\mathbb{R}^d$ is a spectral density for a CCWS random field $X := (X_\nu : \nu \in \mathbb{R}^d)$ if for all $\nu \in \mathbb{R}^d$,

$$\gamma(\nu) = EX_\nu\overline{X}_0 = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\lambda\cdot\nu} f(\lambda)\mathrm{d}m_d(\lambda).$$

Remark 1.3. It will be convenient to write $\gamma(0)$ or X_0 instead of $\gamma(0, 0, \ldots, 0)$ or $X_{(0,0,\ldots,0)}$, where the 0 will be understood as the origin in $\mathbb{R}^d$. It will also be convenient to let $X_1 := X_{(1,1,\ldots,1)}$ for $(1, 1, \ldots, 1) \in \mathbb{R}^d$.

Since an integrable function on $\mathbb{R}^d$ is uniquely determined almost everywhere by its Fourier coefficients, the spectral density function is unique if one disregards sets of Lebesgue measure zero. One can use either the spectral density function or the covariance function to describe their underlying weakly stationary process. Both contain the same information, but are complimentary to one another by expressing this information in different ways [1].

For non-empty sets $Q, S \subset V^d$ where $V = \mathbb{Z}$ or $\mathbb{R}$, $\mathrm{dist}(Q, S) := \min_{q\in Q, s\in S} \|q - s\|$, where $\|\cdot\|$ is the Euclidean distance. When there is an understood fixed positive integer d, $\lambda(\cdot)$ denotes Lebesgue measure on $\mathbb{R}^d$. The L^p norm will be denoted by $\|\cdot\|_p$. The definitions of the linear dependence coefficients to follow are for measuring the dependence of random fields. They are quite similar to the ones defined solely for discrete-indexed random fields which can be found in [3]. They are modified here to account for continous-indexed random fields.

Definition 1.4. Let $X := (X_\nu : \nu \in V^d)$ be a CCWS random field. Let $m(\cdot)$ denote Lebesgue measure if $V = \mathbb{R}$ and the counting measure if $V = \mathbb{Z}$. For any non-empty, disjoint, bounded Borel sets $Q, S \subset V^d$, define the number

$$\mathcal{R}(Q, S) = \sup \frac{\left|EU\overline{W}\right|}{\|U\|_2\|W\|_2}, \tag{1.1}$$

where the supremum is taken over all pairs of complex-valued random variables U and W of the form

$$U = \int_Q j(\nu)X_\nu \mathrm{d}m(\nu) \quad \text{and} \quad W = \int_S j(\nu)X_\nu \mathrm{d}m(\nu),$$

where $j(\nu)$ is a bounded, complex valued Borel function. In (1.1) and the equations below, 0/0 will be interpreted as 0. Note that if $V = \mathbb{Z}$, then the integrals above and below will be sums (since $m(\cdot)$ is a counting measure).

For each $s \in V_+$, define

$$q(X, s) = q(s) := \sup \frac{\left| E\left(\int_Q X_\nu \mathrm{d}m(\nu)\right)\left(\overline{\int_S X_\nu \mathrm{d}m(\nu)}\right)\right|}{\left\|\int_Q X_\nu \mathrm{d}m(\nu)\right\|_2 \left\|\int_S X_\nu \mathrm{d}m(\nu)\right\|_2}, \tag{1.2}$$

$$r(X, s) = r(s) := \sup \mathcal{R}(Q, S), \tag{1.3}$$

where each supremum is taken over all pairs of non-empty bounded Borel sets Q and $S \subset V^d$ such that

$$\left.\begin{array}{c}\text{there exists } u \in \{1, 2, \ldots, d\} \text{ such that}\\ Q \subset \{(k_1, k_2, \ldots, k_d) \in V^d : k_u \le 0\}\\ S \subset \{(k_1, k_2, \ldots, k_d) \in V^d : k_u \ge s\}\end{array}\right\}. \tag{1.4}$$

Again, for each $s \in V_+$, define

$$q'(X, s) = q'(s) := \sup \frac{\left| E\left(\int_Q X_\nu \mathrm{d}m(\nu)\right)\left(\overline{\int_S X_\nu \mathrm{d}m(\nu)}\right)\right|}{\left\|\int_Q X_\nu \mathrm{d}m(\nu)\right\|_2 \left\|\int_S X_\nu \mathrm{d}m(\nu)\right\|_2}, \tag{1.5}$$

$$r'(X, s) = r'(s) := \sup \mathcal{R}(Q, S), \tag{1.6}$$

$$\zeta(X, s) = \zeta(s) := \sup \frac{\left| E\left(\int_Q X_\nu \mathrm{d}m(\nu)\right)\left(\overline{\int_S X_\nu \mathrm{d}m(\nu)}\right)\right|}{m(Q \cup S)} \tag{1.7}$$

where each supremum is taken over all pairs of non-empty bounded Borel sets Q and $S \subset V^d$ such that

$$\left.\begin{array}{c}\text{there exists } u \in \{1, 2, \ldots, d\} \text{ and non-empty sets}\\ Q_0, S_0 \subset V \text{ with } dist(Q_0, S_0) \ge s \text{ such that}\\ Q \subset \{(k_1, k_2, \ldots, k_d) \in V^d : k_u \in Q_0\}\\ S \subset \{(k_1, k_2, \ldots, k_d) \in V^d : k_u \in S_0\}\end{array}\right\}. \tag{1.8}$$

Finally, for each $s \in V_+$, define

$$q^*(X, s) = q^*(s) := \sup \frac{\left| E\left(\int_Q X_\nu \mathrm{d}m(\nu)\right)\left(\overline{\int_S X_\nu \mathrm{d}m(\nu)}\right)\right|}{\left\|\int_Q X_\nu \mathrm{d}m(\nu)\right\|_2 \left\|\int_S X_\nu \mathrm{d}m(\nu)\right\|_2}, \tag{1.9}$$

$$r^*(X, s) = r^*(s) := \sup \mathcal{R}(Q, S), \tag{1.10}$$

where each supremum is taken over all pairs of non-empty, bounded Borel sets Q and $S \subset V^d$ such that $dist(Q, S) \ge s$.

Cauchy's inequality implies that $r(n) \le r'(n) \le r^*(n) \le 1$ and $q(n) \le q'(n) \le q^*(n) \le 1$. Also, it is easy to see that $q(n) \le r(n)$, $q'(n) \le r'(n)$, and $q^*(n) \le r^*(n)$. It is also worth noting that $\zeta(n) \in [0, \infty]$.

Definition 1.5. For a random field $X := (X_k : k \in \mathbb{Z}^d)$ and any $\mathbf{n} := (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$, let

$$S(X, \mathbf{n}) := \sum_k X_k$$

where the sum is taken over all $k := (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$ such that $1 \le k_i \le n_i$ for each $i = 1, 2, \ldots, d$. Often times, $\mathbf{n}$ takes the form $(n, n, \ldots, n)$ for some $n \in \mathbb{Z}_+$. In this case, the boldface will be dropped so that $S(X, \mathbf{n}) = S(X, n)$. A more general sum over a finite subset $Q \subset \mathbb{Z}^d$ will be denoted

$$S(X, Q) := \sum_{k \in Q} X_k.$$

Definition 1.6. For a random field $X := (X_\nu : \nu \in \mathbb{R}^d)$ and any $\mathbf{a} \in \mathbb{R}_+^d$, let

$$I(X, \mathbf{a}) := \int_{(0,\mathbf{a})} X_\nu \mathrm{d}m(\nu)$$

whenever it exists, where $(0, \mathbf{a}) := \prod_{i=1}^d (0, a_i)$ (the Cartesian product). As in the previous definition, when $\mathbf{a} = (a, a, \ldots, a)$ for some $a \in \mathbb{R}_+$ the boldface will be dropped so that $I(X, \mathbf{a}) = I(X, a)$. A more general integral over any bounded Borel set

$Q \subset \mathbb{R}^d$ will be denoted

$$I(X, Q) := \int_Q X_\nu \mathrm{d}m(\nu)$$

whenever it exists.

2. Random fields of continuous index

For a CCWS random field $Y := (Y_k : k \in \mathbb{Z}^d)$, the condition $\zeta(Y, n) \to 0$ as $n \to \infty$ is sufficient for the existence of a continuous spectral density by Theorem 1.4 in [4]. When the CCWS random field is of continuous index $X := (X_\nu : \nu \in \mathbb{R}^d)$, the condition $\zeta(X, s) \to 0$ does not seem to be sufficient for a continuous spectral density. Integrating X_ν over blocks (translations of $[0, 1]^d$) generates a discrete-indexed random field. The lemmas for discrete-indexed random fields can then be extended to include CCWS random fields indexed by $\mathbb{R}^d$. In turn, these lemmas will lead to the following main result.

Theorem 2.1. *Let $X := (X_\nu : \nu \in \mathbb{R}^d)$ be a non-degenerate, CCWS random field. Suppose that $\zeta(s) \to 0$ as $s \to \infty$, and $r'(A) < 1$ for some $A > 0$. If the function $T : \mathbb{R}^d \to \mathbb{R}$ defined by*

$$T(x) := E\left|\int_{[0,1]^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu} X_\nu \,\mathrm{d}\nu\right|^2$$

is integrable, then X has a nonnegative, continuous spectral density function on $\mathbb{R}^d$.

There is another expression for $T(x)$ that will be used later and it is given by

$$T(x) := \int_{[-1,1]^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu} \left(\prod_{i=1}^d (1 - |\nu_i|)\right) \gamma(\nu)\mathrm{d}\nu. \tag{2.1}$$

These two definitions of $T(x)$ are equal, and in fact,

$$E\left|\int_{[0,L]^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu} X_\nu \,\mathrm{d}\nu\right|^2 = \int_{[-L,L]^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu} \left(\prod_{j=1}^d (L - |\nu_j|)\right) \gamma(\nu)\mathrm{d}\nu, \tag{2.2}$$

for any $L > 0$, and $x \in \mathbb{R}^d$. In one dimension, this can be shown using multivariate calculus with the transformation taking the square with vertices $(0, 0)$, $(0, L)$, $(L, 0)$, and (L, L) to the square with vertices $(0, 0)$, (L, L), $(0, 2L)$, and $(-L, L)$. The result for multi-dimensions can be obtained recursively from one dimension. The full proof and calculation can be found in Appendix A of [9].

The following lemma is a consequence of Lemma 2.8 in [3] and is essentially a restatement of Lemma 1.5 in [4].

Lemma 2.2. *Suppose d is a positive integer. Let $\theta := \{\theta_n\}$ be a non-increasing sequence of real numbers in $[0, 1]$ where $\lim_{n\to\infty} \theta_n < 1$. Then there exists a positive number $A := A(\theta, d)$ such that if $X := (X_k : k \in \mathbb{Z}^d)$ is a CCWS random field with $q'(n) \le \theta_n$ for all $n \ge 1$, then for any finite set $Q \subset \mathbb{Z}^d$ one has that*

$$E\,|S(X, Q)|^2 \le A \cdot \mathit{card}\ Q \cdot E|X_0|^2.$$

The next lemma is an extension of Lemma 2.2 to random fields of continuous index which uses the following notation. For any $a \in \mathbb{R}_+$, let $[\![a]\!]$ denote the greatest integer less than or equal to a. If $\mathbf{a} := (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d_+$ and $k := (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$, then let $(\mathbf{0}, \mathbf{a}) := \prod_{i=1}^d (0, a_i)$ and $\mathbf{a}k := (a_1k_1, a_2k_2, \ldots, a_dk_d)$, i.e. coordinatewise multiplication. From this, define $(-\mathbf{a}, \mathbf{0}) + \mathbf{a}k := \prod_{i=1}^d ((k_i - 1)a_i, k_ia_i)$.

Lemma 2.3. *Suppose d is a positive integer. Let $\theta := \{\theta_n\}$ be a non-increasing sequence of real numbers in $[0, 1]$ where $\lim_{n\to\infty} \theta_n < 1$. Then there exists a positive number $B := B(\theta, d)$ such that if $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field with $q'(n) \le \theta_n$ for all $n \ge 1$, then for any $\mathbf{a} := (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d_+$,*

$$E|I(X, \mathbf{a})|^2 \le B \cdot \left(\prod_{i=1}^d a_i\right) \cdot \|X_0\|_2^2.$$

Proof. Let $\theta_0 := 1$ and define the sequence $\theta' := \{\theta'_n\}$ by $\theta'_n = \theta_{[\![(n-1)/2]\!]}$. Let $A_j := A(\theta', j)$ be the constant obtained from Lemma 2.2 for each $j \in \{1, 2, \ldots, d\}$ and then let $B := \max\{1, A_1, A_2, \ldots, A_d\}$. It will be shown that Lemma 2.3 holds with this B.

Suppose that $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field such that $q'(n) \le \theta_n$ for all $n \ge 1$. Fix any $\mathbf{a} \in \mathbb{R}^d_+$. For each $i = 1, 2, \ldots, d$, define $a'_i := a_i/(1 + [\![a_i]\!])$. Then $a'_i < 1$ for every $i \in \{1, 2, \ldots, d\}$. Let $\mathbf{a}' := (a'_1, a'_2, \ldots, a'_d)$. A simple application of Hölder's inequality and Fubini's Theorem yields

$$
\begin{aligned}
E\left|\int_{(\mathbf{0},\mathbf{a}')} X_\nu \mathrm{d}\nu\right|^2 &\le E\left(\int_{(\mathbf{0},\mathbf{a}')} |X_\nu|\mathrm{d}\nu\right)^2 \\
&\le E\left(\int_{(\mathbf{0},\mathbf{a}')} |X_\nu|^2\mathrm{d}\nu \cdot \int_{(\mathbf{0},\mathbf{a}')} 1^2\mathrm{d}\nu\right) \\
&= \left(\prod_{i=1}^{d} a_i'\right)^2 \cdot \|X_0\|_2^2. \qquad (2.3)
\end{aligned}
$$

From this, the proof is complete in the case when $a_i < 1$ for all $i \in \{1, 2, \dots, d\}$ (since $a_i = a_i'$ in this case). Now assume $a_i \ge 1$ for at least one i.

Let $Q := \{i \in \{1, 2, \dots, d\} : a_i \ge 1\}$. Without loss of generality, one can assume that $Q = \{1, \dots, j\}$ for some $j \in \{1, 2, \dots, d\}$ by permuting the indices if necessary. For each $k := (k_1, k_2, \dots, k_j) \in \mathbb{Z}^j$, let $k' := (k_1, \dots, k_j, 1, \dots, 1) \in \mathbb{Z}^d$. With this notation in place, let $0' := (0, \dots, 0, 1, \dots, 1)$ where there are j 0's and $d - j$ 1's. Now, for $k \in \mathbb{Z}^j$, define $Y_k := \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'k'} X_\nu \mathrm{d}\nu$. Then $Y := (Y_k : k \in \mathbb{Z}^j)$ is a discrete parameter random field. Since X is complex and centered, Y is complex and Fubini gives the fact that Y is centered. The weak stationarity of Y will be obtained by using the weak stationarity of X and a few applications of Fubini's theorem. For $h, k \in \mathbb{Z}^j$ (let h' be defined as k' is above and note that $h' - k' = (h - k)' - 0'$),

$$
\begin{aligned}
EY_h\overline{Y}_k &= E\left(\int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'h'} X_\nu \mathrm{d}\nu \cdot \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'k'} \overline{X}_\xi \mathrm{d}\xi\right) \\
&= \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'h'} \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'k'} E\left(X_\nu \overline{X}_\xi\right) \mathrm{d}\xi \mathrm{d}\nu \\
&= \int_{(-\mathbf{a}',\mathbf{0})} \int_{(-\mathbf{a}',\mathbf{0})} E\left(X_{\nu+\mathbf{a}'h'} \overline{X}_{\xi+\mathbf{a}'k'}\right) \mathrm{d}\xi \mathrm{d}\nu \\
&= \int_{(-\mathbf{a}',\mathbf{0})} \int_{(-\mathbf{a}',\mathbf{0})} E\left(X_{\nu+\mathbf{a}'(h'-k')} \overline{X}_{\xi}\right) \mathrm{d}\xi \mathrm{d}\nu \\
&= \int_{(-\mathbf{a}',\mathbf{0})} \int_{(-\mathbf{a}',\mathbf{0})} E\left(X_{\nu+\mathbf{a}'[(h-k)'-0']} \overline{X}_{\xi}\right) \mathrm{d}\xi \mathrm{d}\nu \\
&= \int_{(-\mathbf{a}',\mathbf{0})} \int_{(-\mathbf{a}',\mathbf{0})} E\left(X_{\nu+\mathbf{a}'(h-k)'} \overline{X}_{\xi+\mathbf{a}'0'}\right) \mathrm{d}\xi \mathrm{d}\nu \\
&= \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'(h-k)'} \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'0'} E\left(X_\nu \overline{X}_\xi\right) \mathrm{d}\xi \mathrm{d}\nu \\
&= E\left(\int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'(h-k)'} X_\nu \mathrm{d}\nu \cdot \int_{(-\mathbf{a}',\mathbf{0})+\mathbf{a}'0'} \overline{X}_\xi \mathrm{d}\xi\right) \\
&= EY_{h-k}\overline{Y}_0,
\end{aligned}
$$

and hence, Y is weakly stationary. Since Y is weakly stationary,

$$
\|Y_0\|_2^2 = \|Y_1\|_2^2 = E\left|\int_{(0,\mathbf{a}')} X_\nu \mathrm{d}\nu\right|^2,
$$

(Remark 1.3) and therefore (2.3) yields

$$
\|Y_0\|_2^2 \le \left(\prod_{i=1}^{d} a_i'\right)^2 \cdot \|X_0\|_2^2. \qquad (2.4)
$$

Notice that with the way Y is defined, (1.5) and the fact that $1/2 \le a_i' < 1$ for each $i \in Q$ gives $q'(Y, n) \le q'(X, (n-1)/2)$ for all $n \ge 2$. Since $q'(X, (n-1)/2) \le \theta_{[\![(n-1)/2]\!]}$ for all $n \ge 2$, $q'(Y, n) \le \theta_n'$ for all $n \ge 1$. Let $[\![\tilde{\mathbf{a}}]\!] + \mathbf{1} := ([\![a_1]\!] + 1, [\![a_2]\!] + 1, \dots, [\![a_j]\!] + 1)$, and recall that $a_i = a_i'([\![a_i]\!] + 1)$. Apply Lemma 2.2 and (2.4) above to get

$$
\begin{aligned}
E|I(X, \mathbf{a})|^2 &= E\left|S(Y, [\![\tilde{\mathbf{a}}]\!] + \mathbf{1})\right|^2 \\
&\le A_j \cdot \prod_{i=1}^{j}([\![a_i]\!] + 1) \cdot \|Y_0\|_2^2
\end{aligned}
$$

$$= A_j \cdot \prod_{i=1}^{d}([\![a_i]\!] + 1) \cdot \|Y_0\|_2^2$$

$$\leq B \cdot \prod_{i=1}^{d}([\![a_i]\!] + 1) \cdot \left(\prod_{i=1}^{d} a_i'\right)^2 \cdot \|X_0\|_2^2$$

$$= B \cdot \left(\prod_{i=1}^{d} a_i'([\![a_i]\!] + 1)\right) \cdot \prod_{i=1}^{d} a_i' \cdot \|X_0\|_2^2$$

$$= B \cdot \left(\prod_{i=1}^{d} a_i\right) \cdot \prod_{i=1}^{d} a_i' \cdot \|X_0\|_2^2$$

$$\leq B \cdot \left(\prod_{i=1}^{d} a_i\right) \cdot \|X_0\|_2^2.$$

Thus, the proof of Lemma 2.3 is complete. □

A slightly modified version of the discrete-indexed random field Y in the proof of Lemma 2.3 will be used in the rest of this article. Define $Y := (Y_k : k \in \mathbb{Z}^d)$ by

$$Y_k = \int_{(-1,0)^d+k} X_\nu \, \mathrm{d}\nu. \tag{2.5}$$

By a calculation in the proof of Lemma 2.3 (with a' replaced by $(1, 1, \ldots, 1)$), Y is a CCWS random field. With the definition in (2.5), notice that $S(Y, \mathbf{n}) = I(X, \mathbf{n})$ and $\zeta(Y, n) \leq \zeta(X, n-1)$ for $n \geq 2$ (recall (1.7)). If $\zeta(X, n) \to 0$, these properties and Lemma 2.8 in [4] imply that $\lim_{a\to\infty} [\![a]\!]^{-d} E|I(X, [\![a]\!])|^2$ exists. This limit should hold for $a^{-d}E|I(X, a)|^2$, so it needs to be shown that

$$|a^{-d}E|I(X, a)|^2 - [\![a]\!]^{-d} E|I(X, [\![a]\!])|^2| \to 0 \quad \text{as } a \to \infty. \tag{2.6}$$

The following lemma will help obtain (2.6).

Lemma 2.4. *Suppose that d is a positive integer, $\theta := \{\theta_n\}$ is a non-increasing sequence of numbers in $[0, 1]$ such that $\lim_{n\to\infty} \theta_n < 1$, and $B := B(\theta, d)$ is the constant from Lemma 2.3. If $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field with $q'(n) \leq \theta_n$ for all $n \geq 1$, then for any $a \in \mathbb{R}_+$*

$$\left|E|I(X, a)|^2 - E|I(X, [\![a]\!])|^2\right| \leq 2da^{d-1/2}B\|X_0\|_2^2. \tag{2.7}$$

Proof. First, use Hölder's inequality and Lemma 2.3 to get

$$\|I(X, a) - I(X, [\![a]\!])\|_2 = \left\|\int_{(0,a)^d} X_\nu \mathrm{d}\nu - \int_{(0,[\![a]\!])^d} X_\nu \mathrm{d}\nu\right\|_2$$

$$= \left\|\int_{(0,a)} \cdots \int_{(0,a)} X_\nu \mathrm{d}\nu - \int_{(0,[\![a]\!])} \cdots \int_{(0,[\![a]\!])} X_\nu \mathrm{d}\nu\right\|_2$$

$$= \left\|\sum_{k=1}^{d}\left(\int_{(0,a)} \cdots \int_{(0,a)} \left(\int_{(0,[\![a]\!])} \cdots \int_{(0,[\![a]\!])} X_\nu \mathrm{d}\nu_1 \ldots \mathrm{d}\nu_{k-1}\right) \mathrm{d}\nu_k \ldots \mathrm{d}\nu_d \right.\right.$$

$$\left.\left. - \int_{(0,a)} \cdots \int_{(0,a)} \left(\int_{(0,[\![a]\!])} \cdots \int_{(0,[\![a]\!])} X_\nu \mathrm{d}\nu_1 \ldots \mathrm{d}\nu_k\right) \mathrm{d}\nu_{k+1} \ldots \mathrm{d}\nu_d\right)\right\|_2$$

$$= \left\|\sum_{k=1}^{d} \int_{(0,a)} \cdots \int_{(0,a)} \int_{([\![a]\!],a)} \int_{(0,[\![a]\!])} \cdots \int_{(0,[\![a]\!])} X_\nu \mathrm{d}\nu_1 \ldots \mathrm{d}\nu_{k-1}\mathrm{d}\nu_k\mathrm{d}\nu_{k+1} \ldots \mathrm{d}\nu_d\right\|_2$$

$$\leq \sum_{k=1}^{d} \left\|\int_{(0,a)} \cdots \int_{(0,a)} \int_{([\![a]\!],a)} \int_{(0,[\![a]\!])} \cdots \int_{(0,[\![a]\!])} X_\nu \mathrm{d}\nu_1 \ldots \mathrm{d}\nu_{k-1}\mathrm{d}\nu_k\mathrm{d}\nu_{k+1} \ldots \mathrm{d}\nu_d\right\|_2$$

$$\leq \sum_{k=1}^{d} (a^{d-k}(a - [\![a]\!])[\![a]\!]^{k-1} B\|X_0\|_2^2)^{1/2}$$

$$\leq da^{(d-1)/2}B^{1/2}\|X_0\|_2.$$

Now use the reverse triangle inequality, the result above, and Lemma 2.3 to get

$$\begin{aligned}\left|E|I(X,a)|^2 - E|I(X,[\![a]\!])|^2\right| &= \left|\|I(X,a)\|_2^2 - \|I(X,[\![a]\!])\|_2^2\right| \\ &= |\|I(X,a)\|_2 - \|I(X,[\![a]\!])\|_2| \cdot |\|I(X,a)\|_2 + \|I(X,[\![a]\!])\|_2| \\ &\le \|I(X,a) - I(X,[\![a]\!])\|_2 \cdot 2a^{d/2}B^{1/2}\|X_0\|_2 \\ &\le 2da^{d-1/2}B\|X_0\|_2^2.\end{aligned}$$

Thus, (2.7) holds and the proof is complete. □

Lemma 2.5. *Suppose that $\theta := \{\theta_n\}$ and $z := \{z_n\}$ are non-increasing sequences in $[0, 1]$ and $[0, \infty]$ respectively such that $\lim_{n\to\infty} \theta_n < 1$ and $\lim_{n\to\infty} z_n = 0$. If $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field with $q'(n) \le \theta_n$ and $\zeta(n) \le z_n$ for all $n \ge 1$, then $\lim_{a\to\infty} a^{-d}E|I(X, a)|^2$ exists in $[0, \infty)$.*

This is an extension of Lemma 2.8 in [4] for index sets $\mathbb{R}^d$ with the added condition $q'(n) \le \theta_n$ for all n, and $S(X, n)$ replaced with $I(X, a)$.

Proof. The proof is trivial in the degenerate case, so assume that $\|X_0\|_2^2 > 0$. Let $B := B(\theta, d)$ be the constant from Lemma 2.3. Use Lemma 2.4 and divide both sides of (2.7) by a^d to get

$$\left|a^{-d}E|I(X,a)|^2 - a^{-d}E|I(X,[\![a]\!])|^2\right| \le \frac{2dB\|X_0\|_2^2}{a^{1/2}}. \tag{2.8}$$

Use Lemma 2.3 to get that

$$\begin{aligned}\left|a^{-d}E|I(X,[\![a]\!])|^2 - [\![a]\!]^{-d}E|I(X,[\![a]\!])|^2\right| &= [\![a]\!]^{-d}E|I(X,[\![a]\!])|^2\left|\frac{[\![a]\!]^d}{a^d} - 1\right| \\ &\le \left|\frac{[\![a]\!]^d}{a^d} - 1\right| \cdot B\|X_0\|_2^2.\end{aligned} \tag{2.9}$$

For any $\varepsilon > 0$, one can find an $L > 0$ large enough so that for any $a \ge L$, both of the following hold:

$$\frac{2dB\|X_0\|_2^2}{a^{1/2}} < \frac{\varepsilon}{2}, \tag{2.10}$$

$$\left|\frac{[\![a]\!]^d}{a^d} - 1\right| < \frac{\varepsilon}{2B\|X_0\|_2^2}. \tag{2.11}$$

Use (2.8)–(2.11) with the triangle inequality to get

$$|a^{-d}E|I(X,a)|^2 - [\![a]\!]^{-d}E|I(X,[\![a]\!])|^2| < \varepsilon \tag{2.12}$$

for any $a \ge L$, which confirms (2.6) since ε is arbitrary. Since $\zeta(Y, n) \le \zeta(X, n-1)$ for $n \ge 2$ (refer to (2.5)), then $\zeta(Y, n) \le z_{n-1}$ for all $n \ge 2$. Lemma 2.8 in [4] implies that $\lim_{a\to\infty}[\![a]\!]^{-d}E|I(X,[\![a]\!])|^2$ exists in $[0, \infty)$ since $z_n \to 0$ and $S(Y, [\![a]\!]) = I(X, [\![a]\!])$. This and (2.6) imply that $\lim_{a\to\infty} a^{-d}E|I(X, a)|^2$ exists, and therefore the proof is complete. □

Definition 2.6. Suppose d is a positive integer and $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field such that $\zeta(n) \to 0$ as $n \to \infty$. For each $a \in \mathbb{R}_+$, define $F(X, a) := a^{-d}E|I(X, a)|^2$ and notice that this is real and nonnegative. Now, in reference to Lemma 2.3, define $F(X) := \lim_{a\to\infty} F(X, a)$.

$$\text{For the random field } Y, \text{ it will be understood that } F(Y) = \lim_{n\to\infty} F(Y,n) = \lim_{n\to\infty} n^{-d}E|S(Y,n)|^2.$$

Lemma 2.7. *Suppose that $\theta := \{\theta_n\}$ and $z := \{z_n\}$ are non-increasing sequences in $[0, 1]$ and $[0, \infty]$ respectively such that $\lim_{n\to\infty} \theta_n < 1$ and $\lim_{n\to\infty} z_n = 0$. Then for any given $\varepsilon > 0$, there exists an $L := L(\varepsilon, \theta, z) > 0$ such that if $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field with $E|X_0|^2 \le 1$, $q'_c(n) \le \theta_n$ and $\zeta_c(n) \le z_n$ for all $n \ge 1$, then $|F(X) - F(X, a)| \le \varepsilon$ for all $a \ge L$.*

This is an extension of Lemma 2.10 in [4] for index sets $\mathbb{R}^d$ with the added condition $q'(n) \le \theta_n$ for all n.

Proof. Let $\varepsilon > 0$ be fixed and arbitrary. Define the sequence $z'_n := z_{n-1}$ with $z_0 = \infty$. Let $L_1 := L_1(z'_n, \varepsilon/2)$ be the constant from Lemma 2.10 in [4], and let $B := B(\theta, d)$ be the constant from Lemma 2.3. Choose $L_2 > 0$ large enough so that for all $a \geq L_2$, both of the following hold:

$$\frac{2dB}{a^{1/2}} < \frac{\varepsilon}{4}, \tag{2.13}$$

$$\left| \frac{[\![a]\!]^d}{a^d} - 1 \right| < \frac{\varepsilon}{4B}. \tag{2.14}$$

If Y is defined as it is in (2.5), then $\|Y_0\|_2^2 \leq 1$ by (2.3) and the fact that $\|X_0\|_2^2 \leq 1$. Since both $F(Y)$ and $F(X)$ exist, they must be equal (since $F(Y, [\![a]\!]) = F(X, [\![a]\!])$). The definition of L_1 gives $|F(Y) - F(Y, [\![a]\!])| < \varepsilon/2$ for all $a \geq L_1$, which is the same as $|F(X) - F(X, [\![a]\!])| < \varepsilon/2$. Since $\|X_0\|_2^2 \leq 1$, (2.8) and (2.9), and the triangle inequality together with (2.13) and (2.14) give $|F(X, [\![a]\!]) - F(X, a)| < \varepsilon/2$, for all $a \geq L_2$. If $L := \max\{L_1, L_2\}$, then the triangle inequality yields $|F(X) - F(X, a)| < \varepsilon$ for all $a \geq L$. $\square$

3. The random field $X^{\langle x \rangle}$

Given the random field $X := (X_\nu : \nu \in \mathbb{R}^d)$ and any $x \in \mathbb{R}^d$, define the random field

$$X^{\langle x \rangle} := (X_\nu^{\langle x \rangle} : \nu \in \mathbb{R}^d) \quad \text{where } X_\nu^{\langle x \rangle} := \mathrm{e}^{-\mathrm{i}x\cdot\nu} X_\nu. \tag{3.1}$$

Also, define the random field

$$Y^{\langle x \rangle} := (Y_k^{\langle x \rangle}, k \in \mathbb{Z}^d) \quad \text{where } Y_k^{\langle x \rangle} := \int_{(-1,0)^d + k} X_\nu^{\langle x \rangle} \mathrm{d}\nu. \tag{3.2}$$

Observe that with this definition in place, $\|Y_1^{\langle x \rangle}\|_2^2 = T(x)$ (refer to Remark 1.3 and Theorem 2.1).

Since X is CCWS, an elementary calculation will show that the random field $X^{\langle x \rangle}$ is CCWS. Following the same argument that is in the proof of Lemma 2.3, $Y^{\langle x \rangle}$ is also CCWS. Lemma 2.4 in [4] had the immediate consequence $\zeta(Y^{\langle x \rangle}, n) \leq 16\zeta(Y, n)$. The analogous consequence $\zeta(X^{\langle x \rangle}, s) \leq 16\zeta(X, s)$ will follow from the next lemma which extends Lemma 2.4 from [4] to random fields indexed by $\mathbb{R}^d$. Then, $\zeta(X^{\langle x \rangle}, s) \to 0$ as $s \to \infty$ whenever $\zeta(X, s) \to 0$ as $s \to \infty$. In (1.7), the coefficient ζ was defined for CCWS random fields only. The same definition will be adapted verbatim for the random fields in Lemma 3.1.

Lemma 3.1. *Suppose d is a positive integer, and $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a centered and complex (not necessarily weakly stationary) random field with $E|X_\nu|^2 < \infty$ for all $\nu \in \mathbb{R}^d$ and $\int_B \|X_\nu\|_2 \mathrm{d}\nu < \infty$ for each bounded Borel set $B \subset \mathbb{R}^d$. Suppose s is a positive real number with $\zeta(X, s) < \infty$, and that Q and S are nonempty, disjoint, bounded Borel subsets of $\mathbb{R}^d$ satisfying (1.8).*

If $a(\cdot)$ is a Borel function on $Q \cup S$ such that $a(\nu) \in [0, 1]$ for all $\nu \in Q \cup S$, then

$$\left| E\left(\int_Q a(\nu) X_\nu \mathrm{d}\nu \right) \left(\int_S \overline{a(\nu) X_\nu} \mathrm{d}\nu \right) \right| \leq \zeta(X, s) \lambda(Q \cup S). \tag{3.3}$$

If $c(\cdot)$ is a complex valued Borel function on $Q \cup S$ with $|c(\nu)| \leq 1$ for all $\nu \in Q \cup S$, then

$$\left| E\left(\int_Q c(\nu) X_\nu \mathrm{d}\nu \right) \left(\int_S \overline{c(\nu) X_\nu} \mathrm{d}\nu \right) \right| \leq 16\zeta(X, s) \lambda(Q \cup S). \tag{3.4}$$

Proof. Let $a : Q \cup S \to [0, 1]$ be an arbitrary Borel function. For each positive integer L, partition Q into $\{Q_0^{(L)}, Q_1^{(L)}, \ldots, Q_L^{(L)}\}$ such that $Q_j^{(L)} := \{\nu \in Q : a(\nu) \in [j/L, (j+1)/L)\}$. Partition S accordingly. Since $a(\nu) \in [0, 1]$, $Q_L^{(L)} = \{\nu \in Q : a(\nu) = 1\}$. For each positive integer L, let $V_L = L^{-1} \sum_{j=1}^L \int_{Q_j^{(L)}} j X_\nu \mathrm{d}\nu$ and $W_L = L^{-1} \sum_{j=1}^L \int_{S_j^{(L)}} j X_\nu \mathrm{d}\nu$ (note that V_L and W_L do not change if $\sum_{j=1}^L$ is replaced by $\sum_{j=0}^L$). Then by the fact that $a(\nu) - j/L \geq 0$ on $Q_j^{(L)}$ and using an integral version of Minkowski's inequality,

$$\begin{aligned}
\left\| \int_Q a(\nu) X_\nu \mathrm{d}\nu - V_L \right\|_2 &= \left\| \int_Q a(\nu) X_\nu \mathrm{d}\nu - \sum_{j=0}^L \int_{Q_j^{(L)}} \frac{j}{L} X_\nu \mathrm{d}\nu \right\|_2 \\
&= \left\| \sum_{j=0}^L \int_{Q_j^{(L)}} \left(a(\nu) - \frac{j}{L} \right) X_\nu \mathrm{d}\nu \right\|_2 \\
&\leq \sum_{j=0}^L \left\| \int_{Q_j^{(L)}} \left(a(\nu) - \frac{j}{L} \right) X_\nu \mathrm{d}\nu \right\|_2
\end{aligned}$$

$$\leq \sum_{j=0}^{L} \int_{Q_j^{(L)}} \left(a(\nu) - \frac{j}{L}\right) \|X_\nu\|_2 \, \mathrm{d}\nu$$

$$\leq \sum_{j=0}^{L} \int_{Q_j^{(L)}} \frac{1}{L} \|X_\nu\|_2 \mathrm{d}\nu$$

$$= \frac{1}{L} \int_{Q} \|X_\nu\|_2 \, \mathrm{d}\nu.$$

Thus, $\| \int_Q a(\nu)X_\nu \mathrm{d}\nu - V_L\|_2 \to 0$ as $L \to \infty$. Analogously, $\| \int_S a(\nu)X_\nu \mathrm{d}\nu - W_L\|_2 \to 0$ as $L \to \infty$. With this, it can be easily shown that $EV_LW_L \to E\left(\int_Q a(\nu)X_\nu \mathrm{d}\nu \int_S a(\nu)X_\nu \mathrm{d}\nu\right)$ as $L \to \infty$. Hence, it suffices to show that $|EV_L\overline{W_L}| \leq \zeta(X, s)\lambda(Q \cup S)$. Define $Q^L(j) := Q_j^{(L)} \cup Q_{j+1}^{(L)} \cup \cdots \cup Q_L^{(L)}$ and $S^L(j) := S_j^{(L)} \cup S_{j+1}^{(L)} \cup \cdots \cup S_L^{(L)}$. Then $V_L = L^{-1}\sum_{j=1}^{L} \int_{Q^L(j)} X_\nu \mathrm{d}\nu$ and $W_L = L^{-1}\sum_{j=1}^{L} \int_{S^L(j)} X_\nu \mathrm{d}\nu$. Thus,

$$\begin{aligned}
\left|EV_L\overline{W_L}\right| &= \left|E\left(L^{-2}\sum_{j=1}^{L}\sum_{k=1}^{L} \int_{Q^L(j)} X_\nu \mathrm{d}\nu \int_{S^L(k)} \overline{X_\nu}\mathrm{d}\nu\right)\right| \\
&\leq L^{-2}\sum_{j=1}^{L}\sum_{k=1}^{L} \left|E\left(\int_{Q^L(j)} X_\nu \mathrm{d}\nu \int_{S^L(k)} \overline{X_\nu}\mathrm{d}\nu\right)\right| \\
&\leq L^{-2}\sum_{j=1}^{L}\sum_{k=1}^{L} \zeta(X, s)\lambda(Q^L(j) \cup S^L(k)) \\
&\leq \zeta(X, s)\lambda(Q \cup S),
\end{aligned}$$

and (3.3) holds.

Any complex, Borel function c on $Q \cup S$ with $|c(\nu)| \leq 1$ for all $\nu \in Q \cup S$ can be represented by $a_1(\nu) - a_2(\nu) + \mathrm{i}a_3(\nu) - \mathrm{i}a_4(\nu)$ where $a_j : Q \cup S \to [0, 1]$ is a Borel function for each $j \in \{1, 2, 3, 4\}$. Using this representation and (3.3), (3.4) holds with a simple calculation. □

Since (3.4) holds for any non-empty, disjoint, bounded Borel subsets Q and S of $\mathbb{R}^d$, and any complex valued Borel function $c(\nu)$ such that $|c(\nu)| \leq 1$, (1.7) ensures that

$$\zeta(X^{\langle x\rangle}, s) \leq 16\zeta(X, s) \quad \text{for all } s > 0 \text{ and } x \in \mathbb{R}^d. \tag{3.5}$$

Notice also, that whenever $j(\nu)$ is a bounded, complex, Borel function on $\mathbb{R}^d$, then so is $\mathrm{e}^{\mathrm{i}x\cdot\nu}j(\nu)$. This, along with the definition in (1.6), implies that

$$r'(X^{\langle x\rangle}, s) = r'(X, s) \quad \text{for all } s > 0 \text{ and } x \in \mathbb{R}^d. \tag{3.6}$$

Theorem 3.2. *Suppose that $\theta := \{\theta_n\}$ and $z := \{z_n\}$ are non-increasing sequences in $[0, 1]$ and $[0, \infty]$ respectively such that $\lim_{n\to\infty} \theta_n < 1$ and $\lim_{n\to\infty} z_n = 0$. Then there exists a positive constant $A := A(\theta)$ and a constant $L := L(\varepsilon, \theta, z)$ for each $\varepsilon > 0$ such that the following statement holds. If $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field with $\|X_0\|_2^2 \leq 1$, $r'(n) \leq \theta_n$ and $\zeta(n) \leq z_n$ for all $n \geq 1$, and the function $T : \mathbb{R}^d \to [0, \infty)$ defined by*

$$T(x) := E\left|\int_{(0,1)^d} X_\nu^{\langle x\rangle} \, \mathrm{d}\nu\right|^2$$

is integrable, then each of the following are true:

(a) *For all $x \in \mathbb{R}^d$, $f(x) := \lim_{a\to\infty} F(X^{\langle x\rangle}, a)$ exists in $[0, \infty)$.*
(b) *For any $\varepsilon > 0$, and any $x \in \mathbb{R}^d$, one has that $\left|f(x) - F(X^{\langle x\rangle}, a)\right| \leq \varepsilon$ for every $a \geq L$.*
(c) *The function f is uniformly continuous on $\mathbb{R}^d$.*
(d) *The function f is integrable, and in particular, $f(x) \leq A \cdot T(x)$ for all $x \in \mathbb{R}^d$.*

Proof. Define $z' := \{z'_n\}$, where $z'_n = 16z_n$ and $\theta' := \{\theta'_n\}$, where $\theta'_1 = 1$ and $\theta'_n = \theta_{n-1}$ for $n \geq 2$. Then $\lim_{n\to\infty} z'_n = 0$ and $\lim_{n\to\infty} \theta'_n < 1$ by assumption. For each $\varepsilon > 0$ define $L := L(\varepsilon, \theta, z')$ as the constant from Lemma 2.7. Define $A := A(\theta', d)$ as the constant from Lemma 2.2. These will be the constants for parts (b) and (d).

Suppose $X := (X_\nu : \nu \in \mathbb{R}^d)$ is a CCWS random field such that $E|X_0|^2 \leq 1$, $r'(n) \leq \theta_n$, and $\zeta(n) \leq z_n$ for all $n \geq 1$. For any $x \in \mathbb{R}^d$, $q'(X^{\langle x\rangle}, n) \leq r'(X, n)$ for every integer $n \geq 1$. Apply Lemma 2.5 to the random field $X^{\langle x\rangle}$ using (3.5) and (3.6) to get that $f(x) := \lim_{a\to\infty} F(X^{\langle x\rangle}, a)$ exists in $[0, \infty)$. Therefore (a) holds.

Fix any $\varepsilon > 0$. Lemma 2.7 implies that for each $x \in \mathbb{R}^d$, $|f(x) - F(X^{\langle x\rangle}, a)| < \varepsilon$ for all $a \geq L$. Thus, (b) holds with this constant L.

Suppose $\varepsilon > 0$ and let $L := L(\varepsilon/3, \theta, z)$ be the constant obtained from part (b). Then for every $x \in \mathbb{R}^d$ and for every $a \geq L$, $|f(x) - F(X^{\langle x\rangle}, a)| \leq \varepsilon/3$. Since the function $F(X^{\langle x\rangle}, L)$ is uniformly continuous on $\mathbb{R}^d$ (shown in Lemma A.2 in Appendix A of [9]) let $\delta > 0$ be small enough so that $|F(X^{\langle x\rangle}, L) - F(X^{<y>}, L)| \leq \varepsilon/3$ if $\|x - y\| < \delta$. A simple application of the triangle inequality now gives $|f(x) - f(y)| \leq \varepsilon$ whenever $\|x - y\| < \delta$. Thus, $f(x)$ is uniformly continuous on $\mathbb{R}^d$ and (c) holds.

Suppose $x \in \mathbb{R}^d$, and let $Y^{\langle x\rangle}$ be the CCWS random field defined in (3.2). Since $r'(Y^{\langle x\rangle}, n) \leq r'(X^{\langle x\rangle}, n-1) \leq \theta_{n-1} = \theta'_n$ for all $n \geq 2$, Lemma 2.2 implies

$$\begin{aligned}
F(X^{\langle x\rangle}, n) &= n^{-d} E|I(X^{\langle x\rangle}, n)|^2 \\
&= n^{-d} E|S(Y^{\langle x\rangle}, n)|^2 \\
&\leq A \cdot \|Y_0^{\langle x\rangle}\|_2^2 \\
&= A \cdot \|Y_1^{\langle x\rangle}\|_2^2 \\
&= A \cdot T(x).
\end{aligned}$$

Let $n \to \infty$ and then $f(x) \leq A \cdot T(x)$ by part (a). Since $T(x)$ is integrable and $x \in \mathbb{R}^d$ is arbitrary, part (d) holds for this constant A and the proof is complete. $\square$

The function f in Theorem 3.2 is indeed the spectral density for the random field X. Using part (a) of Theorem 3.2 and (2.2), this function can be written

$$\begin{aligned}
f(x) &= \lim_{L\to\infty} F(X^{\langle x\rangle}, L) \\
&= \lim_{L\to\infty} L^{-d} E\left|\int_{(0,L)^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu} X_\nu \mathrm{d}\nu\right|^2 \\
&= \lim_{L\to\infty} L^{-d} \int_{[-L,L]^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu} \left(\prod_{j=1}^{d} (L - |\nu_j|)\right) \gamma(\nu) \mathrm{d}\nu \\
&= \lim_{L\to\infty} \int_{[-L,L]^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu} \left(\prod_{j=1}^{d} \left(1 - \frac{|\nu_j|}{L}\right)\right) \gamma(\nu) \mathrm{d}\nu.
\end{aligned} \tag{3.7}$$

The integrand in (3.7) is dominated by $|\gamma(\nu)|$. Since γ is not known to be integrable, the inversion theorem cannot be used to show that f is the spectral density for the random field X. A CCWS random field $X^{(\rho)}$ for $\rho \in (0, 1)$ with the property that $\gamma(X^{(\rho)}, \nu) = \gamma(X, \nu) \cdot \rho^{\sum|\nu_i|}$ will be constructed and will satisfy Theorem 3.2. The function f_ρ obtained by Theorem 3.2 for $X^{(\rho)}$ will be exactly like (3.7) with $\rho^{\sum|\nu_i|}$ inserted into the integrand. With this, $\gamma(0) \cdot \rho^{\sum|\nu_i|}$ will be an integrable, dominating function and the inversion theorem can be used to show that f_ρ is the spectral density of $X^{(\rho)}$ for each $\rho \in (0, 1)$. Letting $\rho \to 1^-$ and using Lebesgue's Dominated Convergence Theorem will help show f is the spectral density function of the original random field X.

4. The random field $X^{(\rho)}$

For a given $\rho \in (0, 1)$, the random field $X^{(\rho)} := (X_\nu^{(\rho)} : \nu \in \mathbb{R}^d)$ will make use of standard independent Poisson processes with parameter $\lambda := -\ln\rho$. Fix a $\rho \in (0, 1)$. Let $(\Omega^{(\rho)}, \mathcal{F}^{(\rho)}, P^{(\rho)})$ be a large enough probability space (use Theorem 20.4 in [10]) so that for each $n \in \mathbb{N}$ and $j \in \{1, 2, \ldots, d\}$, the families $\tau_{n,j}$ and $\tau'_{n,j}$ of random variables can be defined on $(\Omega^{(\rho)}, \mathcal{F}^{(\rho)}, P^{(\rho)})$ such that all of the random variables in the entire collection are independent of each other and follow an exponential distribution with parameter $-\ln\rho$.

For each $j \in \{1, 2, \ldots, d\}$, define the random sequence $(\ldots, S_{-1}^j, S_0^j, S_1^j, \ldots)$ on $(\Omega^{(\rho)}, \mathcal{F}^{(\rho)}, P^{(\rho)})$ by $S_n^j(\omega') = \sum_{k=1}^{n} \tau_{k,j}(\omega')$ if $n > 0$ and $S_n^j(\omega') = \sum_{k=1}^{-n+1} -\tau'_{k,j}(\omega')$ if $n \leq 0$. Then for all $a \in \mathbb{R}$ and each $j \in \{1, 2, \ldots, d\}$, let

$$N_a^j(\omega') = \max[n : S_n^j(\omega') \leq a] \tag{4.1}$$

[10, pg. 298]. For each $j \in \{1, 2, \ldots, d\}$, $(N_a^j : a \in \mathbb{R})$ is a Poisson process with rate $-\ln(\rho)$. Without loss of generality, assume that for every $\omega' \in \Omega^{(\rho)}$ and every $j \in \{1, 2, \ldots, d\}$, $N_a^j(\omega') \to -\infty$ as $a \to -\infty$ and $N_a^j(\omega') \to \infty$ as $a \to \infty$. Let $r := (r(1), r(2), \ldots, r(d)) \in \mathbb{R}^d$, and define $N_r(\omega') := (N_{r(1)}^1(\omega'), N_{r(2)}^2(\omega'), \ldots, N_{r(d)}^d(\omega'))$ where each $N_{r(j)}^j(\omega')$ for $j \in \{1, 2, \ldots, d\}$ is defined as in (4.1).

Enlarging the probability space $(\Omega, \mathcal{F}, P)$ if necessary, for each $n \in \mathbb{Z}^d$, define the random field W^n by

$$W^n := (W_r^n : r \in \mathbb{R}^d),$$

so that X and all the W^n are independent and identically distributed. For a technical definition of what is meant by "enlarging a probability space", see Section 5 of Appendix A in [9]. For each fixed $\rho \in (0,1)$, define the random field $X^{(\rho)} := (X_r^{(\rho)} : r \in \mathbb{R}^d)$ on the product space $(\mathbf{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P}) := (\Omega \times \Omega^{(\rho)}, \mathcal{F} \times \mathcal{F}^{(\rho)}, P \times P^{(\rho)})$ by

$$X_r^{(\rho)}(\omega, \omega') = W_r^{N_r(\omega')}(\omega).$$

Note that the random field $X^{(\rho)}$ is defined on d-dimensional blocks where each vertex is a d-tuple of points in each of the d Poisson processes. Every block then contains a new, independent copy of X, namely W^{N_r}.

Since there are three probability spaces present, the notation E_P, E_ρ, and $E_{\mathbf{P}}$ will be used to distinguish between taking expected values with respect to the probability spaces $(\Omega, \mathcal{F}, P)$, $(\Omega^{(\rho)}, \mathcal{F}^{(\rho)}, P^{(\rho)})$, and $(\mathbf{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ respectively.

It can be easily seen that $X^{(\rho)}$ has finite second moments and is centered assuming the original random field is both. Showing that $X^{(\rho)}$ is weakly stationary will use the following notation. For $\nu \in \mathbb{R}^d$, define $\nu_\bullet := \sum_{i=1}^d \nu_i$ and $|\nu|_\bullet := \sum_{i=1}^d |\nu_i|$. Let $\mathbf{1}(\cdot)$ denote the indicator function. Also, observe that for $r, s \in \mathbb{R}^d$ and distinct $j, k \in \mathbb{Z}^d$, $E_P W_r^j \overline{W_s^k} = E_P W_r^j \cdot E_P \overline{W_s^k} = 0 \cdot 0 = 0$. Let $r, s \in \mathbb{R}^d$ be arbitrary. Then

$$\begin{aligned}
E_{\mathbf{P}} X_r^{(\rho)} \overline{X_s^{(\rho)}} &= E_{\mathbf{P}} W_r^{N_r} \overline{W_s^{N_s}} \\
&= \sum_{j\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} E_{\mathbf{P}} \left(W_r^{N_r} \mathbf{1}(N_r = j) \overline{W_s^{N_s} \mathbf{1}(N_s = k)} \right) \\
&= \sum_{j\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} E_{\mathbf{P}} \left(W_r^{j} \mathbf{1}(N_r = j) \overline{W_s^{k}} \mathbf{1}(N_s = k) \right) \\
&= \sum_{j\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} E_P W_r^j \overline{W_s^k} \cdot E_\rho \left(\mathbf{1}(N_r = j) \mathbf{1}(N_s = k) \right) \\
&= \sum_{j\in\mathbb{Z}^d} E_P W_r^j \overline{W_s^j} E_\rho (\mathbf{1}(N_r = j \text{ and } N_s = j)) \\
&= \sum_{j\in\mathbb{Z}^d} E_P X_r \overline{X_s} \cdot P^{(\rho)}(N_r = N_s = j) \\
&= \gamma(X, r-s) \cdot P^{(\rho)}(N_r = N_s) \\
&= \gamma(X, r-s) \cdot P^{(\rho)}(N_{r(1)} = N_{s(1)}, N_{r(2)} = N_{s(2)}, \ldots, N_{r(d)} = N_{s(d)}) \\
&= \gamma(X, r-s) \cdot P^{(\rho)}(N_{r(1)} = N_{s(1)}) \cdot \cdots \cdot P^{(\rho)}(N_{r(d)} = N_{s(d)}) \\
&= \gamma(X, r-s) \cdot P^{(\rho)}(N_{r(1)} - N_{s(1)} = 0) \cdot \cdots \cdot P^{(\rho)}(N_{r(d)} - N_{s(d)} = 0) \\
&= \gamma(X, r-s) \prod_{j=1}^d \rho^{|r(j)-s(j)|} \\
&= \gamma(X, r-s) \cdot \rho^{|r-s|_\bullet},
\end{aligned}$$

where the second-to-last equality is done in Billingsley [10] (23.9). Since the covariance function depends only on the difference of the subscripts, $X^{(\rho)}$ is weakly stationary. Hence, $X^{(\rho)}$ is a CCWS random field and $\gamma(X^{(\rho)}, \nu) = \gamma(X, \nu) \cdot \rho^{|\nu|_\bullet}$.

The CCWS random field $X^{(\rho)}$ will now be shown to satisfy Theorem 3.2 (assuming X does). Without loss of generality, assume that $\|X_0^{(\rho)}\|_2^2 = \|X_0\|_2^2 \le 1$ (multiply by a constant). From the construction of $X^{(\rho)}$, it should be intuitively obvious that $\zeta(X^{(\rho)}, s) \le \zeta(X, s)$ and $r'(X^{(\rho)}, s) \le r'(X, s)$, since $X^{(\rho)}$ is more weakly dependent than X (all W^n's are independent), and ζ and r' are linear dependence coefficients. These inequalities are not trivial to show, however, and can be found in Chapter 6 of [9]. Recall that $E_{\mathbf{P}} X_\nu^{(\rho)} \overline{X_0^{(\rho)}} = \gamma(X^{(\rho)}, \nu) = \rho^{|\nu|_\bullet} \gamma(X, \nu)$. Then for each fixed $\rho \in (0,1)$,

$$T^{(\rho)}(x) = \int_{[-1,1]^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu} \left(\prod_{i=1}^d (1 - |\nu_i|) \right) \gamma(X^{(\rho)}, \nu) \mathrm{d}\nu$$

for $x \in \mathbb{R}^d$. If $T^{(\rho)}$ is integrable, then $X^{(\rho)}$ will satisfy Theorem 3.2 assuming the inequalities in the previous paragraph. Some non-standard Fourier analysis techniques will be used to show $T^{(\rho)}$ is integrable. Most of these techniques are variations of those from Chapter 9 in [11] and Chapter 7 in [12]. To simplify the appearance of some calculations ahead, let $\mu_d(\cdot)$ be the normalized Lebesgue measure on $\mathbb{R}^d$ defined by $\mathrm{d}\mu_d(x) = (2\pi)^{-d/2}\mathrm{d}x$. The Fourier transform $\hat{f} : \mathbb{R}^d \to \mathbb{R}$ of a function $f \in L^1(\mathbb{R}^d)$ will be defined by

$$\hat{f}(\nu) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\nu\cdot x} f(x) \mathrm{d}\mu_d(x).$$

This is not standard. In most texts, $\mathrm{e}^{\mathrm{i}\nu\cdot x}$ would be replaced by $\mathrm{e}^{-\mathrm{i}\nu\cdot x}$ in the definition above. The theory is the same and makes the arguments to follow a little easier.

Theorem 4.1. *Suppose $f, g \in L^1(\mathbb{R}^d)$. Then $f * g \in L^1(\mathbb{R}^d)$ and $\widehat{f * g} = \hat{f}\hat{g}$.*

Theorem 4.2. *If $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^1(\mathbb{R}^d)$, then*

$$f(x) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu} \hat{f}(\nu)\mathrm{d}\mu_d(\nu)$$

for almost every $x \in \mathbb{R}^d$. If f is also assumed to be continuous, then the equality holds for all $x \in \mathbb{R}^d$.

Theorem 4.2 is taken from part (c) of Section 7.7 in [12].

For $t, x \in \mathbb{R}^d$, let $|t|_\bullet = \sum_{i=1}^d |t_i|$ and $t \cdot x = \sum_{i=1}^d t_i x_i$. Define $H : \mathbb{R}^d \to \mathbb{R}$ by

$$H(t) = \mathrm{e}^{-|t|_\bullet},$$

and define

$$h_\lambda(x) := \int_{\mathbb{R}^d} H(\lambda t)\mathrm{e}^{\mathrm{i}t\cdot x}\mathrm{d}\mu_d(t) \quad (\lambda > 0),$$

and notice these are even functions. Hence, $h_\lambda(x) = \int_{\mathbb{R}^d} H(\lambda t)\mathrm{e}^{-\mathrm{i}t\cdot x}\mathrm{d}\mu_d(t)$. A simple computation will show

$$h_\lambda(x) = \left(\frac{2}{\pi}\right)^{d/2} \prod_{i=1}^d \frac{\lambda}{\lambda^2 + x_i^2},$$

and therefore, $\int_{\mathbb{R}^d} h_\lambda(x)\mathrm{d}\mu_d(x) = 1$. By Theorem 4.2, $H(\lambda t) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}t\cdot\nu} h_\lambda(\nu)\mathrm{d}\mu_d(\nu)$ for every $t \in \mathbb{R}^d$, and therefore, $H(\lambda t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}t\cdot\nu} h_\lambda(\nu)\mathrm{d}\mu_d(\nu)$ for every $t \in \mathbb{R}^d$ since h_λ is even. Note that this implies $H(\lambda t) = \hat{h}_\lambda(t)$.

Theorem 4.3. *If $f \in L^1(\mathbb{R}^d)$, then*

$$\lim_{\lambda\to 0^+} \|f * h_\lambda - f\|_1 = 0.$$

For $d = 1$, this was done in Theorem 9.10 of [11]. The argument for general d is analogous.

Let $g(\nu) = \mathbf{1}_{[-1,1]^d}(\nu)(\prod_{j=1}^d (1 - |\nu_j|))\gamma(X, \nu)$, so that $T(x) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu} g(\nu)\mathrm{d}\nu$. The function g is bounded with bounded support and therefore is integrable. Define $\mathcal{T}(x) := (2\pi)^{-d/2}T(x)$ and $\mathcal{T}^{(\rho)}(x) := (2\pi)^{-d/2}T^{(\rho)}(x)$. Under the assumption that $T(x)$ is integrable, $\mathcal{T}(x)$ is integrable. Theorem 4.2 implies that $g(x) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\nu}\mathcal{T}(\nu)\mathrm{d}\mu_d(\nu)$ for almost every $x \in \mathbb{R}^d$, and therefore $g(x) = \widehat{\mathcal{T}}(x)$ for almost every $x \in \mathbb{R}^d$. In fact, $g(x) = \widehat{\mathcal{T}}(x)$ for all $x \in \mathbb{R}^d$ since they are continuous and equal almost everywhere.

With $\lambda = -\ln(\rho)$ for $\rho \in (0, 1)$, define $H_\rho(\nu) = H(\lambda\nu) = \rho^{|\nu|_\bullet}$, and let

$$h_\rho(x) = \int_{\mathbb{R}^d} H_\rho(\nu)\mathrm{e}^{-\mathrm{i}x\cdot\nu}\mathrm{d}\mu_d(\nu).$$

Since both H_ρ and h_ρ are continuous, integrable, even functions, Theorem 4.2 implies

$$H_\rho(x) = \int_{\mathbb{R}^d} h_\rho(\nu)\mathrm{e}^{\mathrm{i}x\cdot\nu}\mathrm{d}\mu_d(\nu)$$

for every $x \in \mathbb{R}^d$. Thus, $H_\rho(x) = \hat{h}_\rho(x)$ for every $x \in \mathbb{R}^d$.

Since $\mathcal{T}$ is integrable and h_ρ is integrable for all $\rho \in (0, 1)$, $\mathcal{T} * h_\rho$ is integrable for all $\rho \in (0, 1)$. The equalities above and Theorem 4.1 imply $\widehat{\mathcal{T} * h_\rho} = \widehat{\mathcal{T}} \cdot \hat{h}_\rho = g \cdot H_\rho$. Since $g \cdot H_\rho$ is integrable, Theorem 4.2 implies

$$\begin{aligned}(\mathcal{T} * h_\rho)(x) &= \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu} g(\nu)H_\rho(\nu)\mathrm{d}\mu_d(\nu) \\ &= \int_{[-1,1]^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu} \left(\prod_{j=1}^d (1 - |\nu_j|)\right)\gamma(X, \nu)\rho^{|\nu|_\bullet}\mathrm{d}\mu_d(\nu) \\ &= \mathcal{T}^{(\rho)}(x)\end{aligned}$$

for every $x \in \mathbb{R}^d$. Hence, $\mathcal{T}^{(\rho)}$ is integrable for $\rho \in (0, 1)$ which implies that $T^{(\rho)}$ is also. Thus, $X^{(\rho)}$ satisfies Theorem 3.2 for all $\rho \in (0, 1)$.

Theorem 4.3 implies that $\|\mathcal{T} * h_\rho - \mathcal{T}\|_1 \to 0$ as $\rho \to 1^-$. This is the same as $\|\mathcal{T}^{(\rho)} - \mathcal{T}\|_1 \to 0$ as $\rho \to 1^-$, which in turn implies

$$\|T^{(\rho)} - T\|_1 \to 0 \text{ as } \rho \to 1^-. \tag{4.2}$$

5. Proof of Theorem 2.1

Let $X := (X_\nu : \nu \in \mathbb{R}^d)$ be a non-degenerate, CCWS random field such that $\zeta(s) \to 0$ as $s \to \infty$, and $r'(a) < 1$ for some $a > 0$. Also, suppose that $T(x)$ (defined in Theorem 2.1 and in (2.1)) is integrable. Without loss of generality, assume that $\|X_0\|_2^2 \le 1$ (multiply the field by appropriate constant if needed). Define the non-increasing sequences $\theta := \{\theta_n\}$ and $z := \{z_n\}$ by $\theta_n := r'(X, n)$ and $z_n := \zeta(X, n)$. The CCWS random field $X^{(\rho)}$ from Section 4 satisfies Theorem 3.2 under these two sequences for each $\rho \in (0, 1)$.

The proof is trivial in the degenerate case, so assume that $0 < \|X_0\|_2 \le 1$. Define $X^{(\rho,x)} := (X_\nu^{(\rho,x)} : \nu \in \mathbb{R}^d)$ where $X_\nu^{(\rho,x)} = \mathrm{e}^{-\mathrm{i}x\cdot\nu} X_\nu^{(\rho)}$. Theorem 3.2 implies that for every $x \in \mathbb{R}^d$, both $f_\rho(x) := \lim_{a\to\infty} F(X^{(\rho,x)}, a)$ and $f(x) := \lim_{a\to\infty} F(X^{\langle x\rangle}, a)$ exist, and that the functions f_ρ and f are continuous and integrable. It will now be shown that $f_\rho(x) \to f(x)$ uniformly as $\rho \to 1^-$. It suffices to show that for each $\varepsilon > 0$, there exists a $\rho_1 \in (0, 1)$ such that $|f_\rho(x) - f(x)| \le \varepsilon$ for all $x \in \mathbb{R}^d$ whenever $\rho \in [\rho_1, 1)$.

Fix any $\varepsilon > 0$, and let $L := L(\varepsilon/3, \theta, z)$ be the constant from Theorem 3.2. Then $|f_\rho(x) - F(X^{(\rho,x)}, a)| \le \varepsilon/3$ and $|f(x) - F(X^{\langle x\rangle}, a)| \le \varepsilon/3$ for every $a \ge L$. Let $\rho_1 \in (0, 1)$ be such that $|1 - \rho_1^{dL}| \le \varepsilon/(3(2L)^d\|X_0\|_2^2)$. Now, refer to (2.2), and note that

$$
\begin{aligned}
&\left|F(X^{(\rho,x)}, L) - F(X^{\langle x\rangle}, L)\right| = \left|L^{-d}E|I(X^{(\rho,x)}, L)|^2 - L^{-d}E|I(X^{\langle x\rangle}, L)|^2\right| \\
&= \left|L^{-d}\int_{[-L,L]^d} \mathrm{e}^{-\mathrm{i}x\cdot r}\prod_{i=1}^d (L - |r_i|)\gamma(r)\rho^{|r|_\bullet}\mathrm{d}r - L^{-d}\int_{[-L,L]^d} \mathrm{e}^{-\mathrm{i}x\cdot r}\prod_{i=1}^d (L - |r_i|)\gamma(r)\mathrm{d}r\right| \\
&= \left|\int_{[-L,L]^d} \mathrm{e}^{-\mathrm{i}x\cdot r}\prod_{i=1}^d \left(1 - \frac{|r_i|}{L}\right)\gamma(r)\rho^{|r|_\bullet}\mathrm{d}r - \int_{[-L,L]^d} \mathrm{e}^{-\mathrm{i}x\cdot r}\prod_{i=1}^d \left(1 - \frac{|r_i|}{L}\right)\gamma(r)\mathrm{d}r\right| \\
&= \left|\int_{[-L,L]^d} \mathrm{e}^{-\mathrm{i}x\cdot r}\prod_{i=1}^d \left(1 - \frac{|r_i|}{L}\right)\gamma(r)(\rho^{|r|_\bullet} - 1)\mathrm{d}r\right| \\
&\le \int_{[-L,L]^d} \|X_0\|_2^2|\rho^{dL} - 1|\mathrm{d}r \\
&\le (2L)^d\|X_0\|_2^2\left(\frac{\varepsilon}{3(2L)^d\|X_0\|_2^2}\right) \\
&= \frac{\varepsilon}{3}
\end{aligned}
$$

for all $\rho \in [\rho_1, 1)$ and all $x \in \mathbb{R}^d$. Thus, the triangle inequality shows that for any $\rho \in [\rho_1, 1)$ and any $x \in \mathbb{R}^d$, $|f_\rho(x) - f(x)| \le \varepsilon$. This implies the uniform convergence of f_ρ to f as $\rho \to 1^-$.

By (4.2), for any $\varepsilon > 0$ there exists a $\rho' \in (0, 1)$ such that $\|T^{(\rho)} - T\|_1 < \varepsilon$ for all $\rho \in [\rho', 1)$. Create a sequence $\{\rho_j\}_{j=1}^\infty$ (all in $(0, 1)$) such that $\rho_j \to 1$ as $j \to \infty$ and $\|T - T^{(\rho(j))}\|_1 \le 1/2^j$. Then for any j, $\|T^{(\rho(j))}\|_1 \le \sum_{k=1}^\infty \|T^{(\rho(k))} - T\|_1 + \|T\|_1 \le 1 + \|T\|_1$. Define $G(x) := \sum_{j=1}^\infty |T^{(\rho(j))}(x) - T(x)| + T(x)$. This function is integrable since $\|T^{(\rho(j))} - T\|_1 \le 1/2^j$ and $T(x)$ is integrable and non-negative. For any fixed j, notice that $T^{(\rho(j))}(x) \le |T^{(\rho(j))}(x) - T(x)| + T(x) \le G(x)$. The function $G(x)$ will be a dominating function for all of the $T^{(\rho(j))}$ and T. Let $A := A(\theta, d)$ be the constant from Theorem 3.2. Then by the definition of G, part (d) of Theorem 3.2 implies $f(x) \le A \cdot G(x)$ and $f_{\rho(j)}(x) \le A \cdot G(x)$. Lebesgue's dominated convergence theorem implies $\int_{\mathbb{R}^d} |f_{\rho(j)}(x) - f(x)|\mathrm{d}x \to 0$ as $j \to \infty$. In particular, for all $\nu \in \mathbb{R}^d$,

$$
\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\nu} f_{\rho(j)}(x)\mathrm{d}x \to \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\nu} f(x)\mathrm{d}x \tag{5.1}
$$

as $j \to \infty$. Again, refer to (2.2) and notice that for a given $x \in \mathbb{R}^d$,

$$
\begin{aligned}
f_\rho(x) &= \lim_{L\to\infty} F(X^{(\rho,x)}, L) \\
&= \lim_{L\to\infty} L^{-d}E|I(X^{(\rho,x)}, L)|^2 \\
&= \lim_{L\to\infty} L^{-d}\int_{[-L,L]^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu}\left(\prod_{i=1}^d (L - |\nu_i|)\right)\gamma(\nu)\rho^{|\nu|_\bullet}\mathrm{d}\nu \\
&= \lim_{L\to\infty} \int_{[-L,L]^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu}\prod_{i=1}^d \left(1 - \frac{|\nu_i|}{L}\right)\gamma(\nu)\rho^{|\nu|_\bullet}\mathrm{d}\nu \\
&= \lim_{L\to\infty} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot\nu}\mathbf{1}_{[-L,L]^d}(\nu)\cdot\prod_{j=1}^d \left(1 - \frac{|\nu_j|}{L}\right)\gamma(\nu)\rho^{|\nu|_\bullet}\mathrm{d}\nu.
\end{aligned} \tag{5.2}
$$

For each $\nu \in \mathbb{R}^d$, the integrand in (5.2) converges to $e^{-ix\cdot\nu}\gamma(\nu)\rho^{|\nu|_\bullet}$ as $L \to \infty$ and is dominated by $\gamma(0)\cdot\rho^{|\nu|_\bullet}$. Since $\rho^{|\nu|_\bullet}$ is integrable, Lebesgue's dominated convergence theorem gives $f_\rho(x) = \int_{\mathbb{R}^d} e^{-ix\cdot\nu}\gamma(\nu)\rho^{|\nu|_\bullet}d\nu$. Multiplying both sides of this equation by $(2\pi)^{-d/2}$, and using the fact that both f_ρ and $\gamma(\nu)\rho^{|\nu|_\bullet}$ are continuous and integrable, Theorem 4.2 implies $\gamma(\nu)\rho^{|\nu|_\bullet} = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\nu}f_\rho(x)dx$ for every $\nu \in \mathbb{R}^d$. Since $\gamma(\nu)\rho^{|\nu|_\bullet} \to \gamma(\nu)$ as $\rho \to 1^-$, (5.1) implies that

$$\gamma(\nu) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\nu}f(x)dx.$$

This shows that f is a spectral density for the random field X. Hence, the proof of Theorem 2.1 is complete.

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